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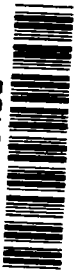
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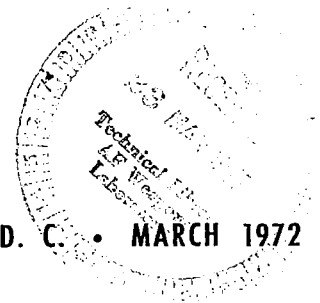
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**FEASIBILITY STUDY
OF SHELL BUCKLING ANALYSIS
USING THE MODIFIED STRUCTURE METHOD**

by Gerald A. Cohen and Raphael T. Haftka

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NATIONAL AERONAUTICS AND SPACE ADMINISTRATION • WASHINGTON, D. C. • MARCH 1972





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| 1. Report No. NASA CR-2008 | 2. Government Accession No. | 3. Recipient's Catalog No. | |
| 4. Title and Subtitle FEASIBILITY STUDY OF SHELL BUCKLING ANALYSIS USING THE MODIFIED STRUCTURE METHOD | | 5. Report Date March 1972 | |
| | | 6. Performing Organization Code | |
| 7. Author(s) Gerald A. Cohen and Raphael T. Haftka | | 8. Performing Organization Report No. | |
| 9. Performing Organization Name and Address Structures Research Associates | | 10. Work Unit No. | |
| | | 11. Contract or Grant No. NAS1-10800 | |
| 12. Sponsoring Agency Name and Address National Aeronautics & Space Administration Washington, DC 20546 | | 13. Type of Report and Period Covered Contractor Report | |
| | | 14. Sponsoring Agency Code | |
| 15. Supplementary Notes | | | |
| 16. Abstract The modified structure method, which is based on Koiter's theory of imperfections, was used to calculate approximate buckling loads of several shells of revolution. The method does not appear to be practical for shells because, in many cases, the prebuckling nonlinearity may be too large to be treated accurately as a small imperfection. | | | |
| 17. Key Words (Suggested by Author(s)) Modified structure method, Koiter's theory of imperfections, nonlinear axisymmetric prestress state, limit load, bifurcation buckling, shells of revolution | | 18. Distribution Statement Unclassified - Unlimited | |
| 19. Security Classif. (of this report) Unclassified | 20. Security Classif. (of this page) Unclassified | 21. No. of Pages 36 | 22. Price* \$3.00 |

*For sale by the National Technical Information Service, Springfield, Virginia 22151

1. Spherical Shells -- Mechanical Properties
 2. Buckling
 3. Shells (Structural Form) -- Mechanical Properties

24 Mar 72

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FEASIBILITY STUDY OF SHELL BUCKLING ANALYSIS

USING THE MODIFIED STRUCTURE METHOD

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SUMMARY

The "modified structure" method for the approximate calculation of nonlinear buckling loads is formulated. In essence, this method consists of viewing the nonlinearity in the prebuckling state as being caused by an imperfection in the load system. The linear solution is an exact solution if the real loads are augmented by a fictitious set of loads. The nonlinear behavior under the real loads is recovered from the linear behavior by considering a load imperfection equal and opposite to the fictitious loads in the context of Koiter's imperfection theory. The method is appropriate only if the fictitious loads are small enough to be treated as an imperfection.

The feasibility of applying this method to shells is examined by treating numerically several cases of shells of revolution under axisymmetric loads. Both bifurcation and limit load buckling are considered. Except for cases with very small nonlinearity, the method yields poor results due to unexpectedly large fictitious loads.

In addition, the growth of error in the first-order load imperfection theory used is evaluated by varying the size of the imperfection for two shallow spherical cap cases (one limit load and one bifurcation case) from zero to the full imperfection implied by the modified structure method. It is shown that the error grows much faster in the bifurcation case than in the limit load case. However, even in the limit load case, the error grows to an unacceptable value (93.6%) when the imperfection reaches the size of the fictitious loads.

It is therefore concluded that, although the method is theoretically sound, in cases with significant nonlinearity the fictitious loads may be too large for its application using first-order load imperfection theory.

INTRODUCTION

Asymptotic methods for the analysis of postbuckling behavior and imperfection sensitivity of structures date back to Koiter's doctoral dissertation (ref. 1) in 1945. It was not, however, until the early sixties that the usefulness of these methods was generally recognized. Though Koiter's method has been successfully used to explain the large discrepancies between experimental and calculated buckling loads, it is still only in the development state as a design tool for predicting buckling loads of engineering structures.

The advantages of Koiter's method over a fully nonlinear analysis of structures are twofold. First, a nonlinear problem is reduced to a small number (usually one or two) of linear problems. Second, Koiter's analysis yields the approximate behavior of a whole class of structures differing from the original by small "imperfections."

Despite these advantages, Koiter's method has been applied only to a relatively small class of problems because of limitations which have restricted its widespread use. In its original form it applies only to structures which have a bifurcation type of buckling. The fact is that limit load buckling is much more common. In addition, prior knowledge of the prebuckling path is necessary to apply the method. The calculation of prebuckling behavior is usually a nonlinear problem. If this nonlinear problem is not much simpler than that of the postbuckling behavior, then most of the advantage of the asymptotic method is lost.

Shells of revolution under axisymmetric loading generally satisfy the above conditions for the application of the method. Buckling is usually of the bifurcation type as the behavior changes from axisymmetric to asymmetric, and even if the prebuckling behavior is nonlinear, it is much simpler to obtain than the exact postbuckling behavior. It is understandable, therefore, that most applications of Koiter's method to shell structures have been to shells of revolution. Examples are given by Koiter (ref. 2), Budiansky, Hutchinson and their students (refs. 3 and 4) for spheres, cylinders and toroidal segments, and by Cohen's (ref. 5) computer program for ring-stiffened orthotropic shells of revolution which have a unique buckling mode.

A way of generalizing Koiter's method to remove the two limitations noted above was recently proposed by Haftka, Mallett and Nachbar (ref. 6) and successfully applied to a number of frame and arch problems (ref. 7). This so-called "modified structure" method is based on treating the nonlinearity of the prebuckling path as a special kind of imperfection. The linear stability analysis, for which standard methods are available for most engineering structures, is viewed not as an approximation to the actual behavior of the structure, but as the exact behavior of the

structure under a modified load system. The difference between the actual and the modified load systems is viewed as a load imperfection. As the linearized stability analysis almost always produces a bifurcation type of buckling, it is possible to retrieve the behavior of the actual structure by using Koiter's method to account for the influence of this load imperfection. This analysis represents very little extra effort compared to the linear stability analysis but gives almost all the information of a costly nonlinear analysis.

The buckling of shells of revolution under asymmetric loading is a problem which may be suitable for the application of the modified structure method. The buckling is usually of the limit load type and even in bifurcation cases the nonlinear prebuckling state is very expensive to calculate. If, however, buckling loads are calculated on the basis of a linearized prebuckling behavior, the modified structure method may improve the accuracy of such a calculation, using it as a first step, with modest additional computational effort. It may be noted here that a computer program for the calculation of these linearized buckling loads has recently been developed under a NASA contract.

A natural way to verify the applicability of the method to shells of revolution is to start with the case of axisymmetric loading. For this case nonlinear solutions are available or may be readily obtained and may be used to evaluate the accuracy and efficiency of the method. The purpose of the present study is to assess the accuracy of the modified structure method for shell structures by actual application of the method to several cases of shells of revolution under axisymmetric loading.

SYMBOLS

| | |
|--------------------|--|
| a, b | first and second postbuckling coefficients |
| c | dimensionless area of ring cross section |
| E | Young's modulus |
| e_1, e_2 | meridional and circumferential linear strain expressions |
| F_x, F_y, F_ϕ | external ring force components per unit of length |
| $F^{(k)}(u, v)$ | work functionals (ref. 9) |
| h | shallow spherical shell rise |
| L_1, L_2 | external shell surface moments per unit of area in the meridional and circumferential directions, respectively |

| | |
|-------------------------|--|
| ℓ | cylindrical shell length |
| \tilde{M}_{12} | modified shear stress couple |
| N_x, N_y, N_ϕ | external ring moment components per unit of length |
| n | circumferential wave number |
| p | pressure distribution associated with normal pressure field $\lambda p(x, y, \phi)$; also pressure applied to spherical cap |
| R | spherical radius |
| r | small circle radius |
| r_a | semimajor axis of spheroid |
| r_b | semiminor axis of spheroid |
| S | effective shell force per unit of circumferential length |
| s, ϕ | meridional and circumferential shell coordinates |
| T_1, T_2 | meridional and circumferential stress resultants |
| \tilde{T}_{12} | modified shear stress resultant |
| T_ϕ | ring hoop stress resultant |
| t | shell thickness |
| u, w | meridional and normal displacement components |
| \underline{u} | shell displacement vector |
| X_1, X_2, X_3 | external shell surface forces per unit of area in meridional, circumferential, and normal directions, respectively |
| x, y | axial and radial coordinates |
| α, β, γ | imperfection functionals |
| Γ | effective γ for limit load case, $2(-a_\alpha)^{1/2}$ |
| Δ | pertaining to a fictitious load |
| δ | first variation |

| | |
|-----------------------------------|--|
| θ_{12} | shear thermal stress |
| $\theta_1, \theta_2, \theta_{12}$ | meridional, circumferential, and shear shell free thermal strains |
| θ_R | circumferential free thermal strain of a ring |
| Λ | shallow spherical shell parameter, $2[3(1 - \nu^2)]^{1/4} (h/t)^{1/2}$ |
| λ | load factor |
| ν | Poisson's ratio |
| ξ | perturbation parameter in imperfection analysis |
| $\bar{\xi}$ | imperfection amplitude |
| ξ, η | axial and radial shell displacements |
| $\chi, -\psi, \omega$ | shell rotations about circumferential, meridional, and normal directions, respectively |
| ω_x, ω_y | ring rotation components |

Generalized field variables and operators:

| | |
|----------------------------|--|
| $H(\epsilon)$ | linear operator relating stress to strain |
| $L_1(u)$ | linear operator representing linear part of strain-displacement relations |
| $L_2(u)$ | quadratic operator representing nonlinear part of strain-displacement relations |
| $L_{11}(u, v)$ | bilinear operator defined by the identity $L_2(u + v) = L_2(u) + 2L_{11}(u, v) + L_2(v)$ |
| $q(\lambda)$ | mechanical load |
| q_0 | dead part of mechanical load at $\lambda = 1$ |
| $q_1(u)$ | linear operator representing live part of mechanical load at $\lambda = 1$ |
| u | displacement |
| u_{01}, u_{01}^i, u_{11} | expansion states defined by equation (5) |
| ϵ | strain |

| | |
|-------------------|--------------------------------------|
| $\theta(\lambda)$ | free thermal strain |
| θ_0 | free thermal strain at $\lambda = 1$ |
| σ | stress |

Subscripts:

| | |
|-------|---|
| c | at bifurcation of perfect structure |
| s | at buckling of imperfect structure |
| 0,1,2 | pertaining to prebuckling, buckling, or postbuckling states, respectively |
| 0() | pertaining to the axisymmetric component of the second-order postbuckling state |

Superscripts:

| | |
|-------------------------|--|
| (1) | $\partial()/\partial\lambda$ |
| (2) | $\partial^2()/\partial\lambda^2$ |
| ($\bar{}$) | pertaining to imperfection at $\bar{\xi} = 1$ |
| (\wedge) | pertaining to linear response at $\lambda = 1$ |
| ()* | evaluated at $\lambda = \lambda_c$ |
| () $^{\circ}$ | $\partial()/\partial\phi$ |

ANALYTICAL FORMULATION

The modified structure method is based upon the idea that the linear approximation to the behavior of a structure under a given load system is also the exact behavior under a different set of loads. The difference between the two sets of loads is treated as a load imperfection in the framework of Koiter's method to correct the buckling load obtained from the linear analysis. This method is appropriate, of course, only if the exact behavior is not so highly nonlinear that the difference between the two sets of loads is too large to be dealt with as an imperfection.

The formulation of the method is naturally divided into two independent parts: (1) the load imperfection analysis, which yields the change in the

behavior of a bifurcating structure due to a small change in the load system, and (2) the determination of the fictitious loads with which the real loads must be augmented in order to enforce a linear prebuckling response. In the modified structure method, it is convenient to think of the "perfect structure" as the structure with the augmented load system (real plus fictitious) applied to it. This structure, which has a linear prebuckling response, is analyzed by conventional numerical methods. The "imperfect structure" is the same structure with only the real loads (the loads of interest) applied to it. Its behavior is approximately recovered from the behavior of the perfect structure through the load imperfection theory, i.e., by treating the negative of the fictitious loads as an imperfection.

Load Imperfection Analysis

Imperfections usually considered in the application of Koiter's method are initial deformations, i.e., geometric. The theory, however, is applicable to other kinds of imperfections, and here it is applied to load imperfections. The imperfection in the load system is viewed as additional sets of mechanical loads $\bar{\xi}\bar{q}$ and free thermal strains $\bar{\xi}\bar{\theta}$, where $\bar{\xi}$ is a scalar imperfection amplitude. Both \bar{q} and $\bar{\theta}$ may depend on the loading intensity λ . The development here is similar to that of reference 8 for geometric imperfections.

The basic equations describing the behavior of an elastic structure with a load imperfection are: the strain displacement relationship

$$\epsilon = L_1(u) + (1/2)L_2(u) \quad (1a)$$

the constitutive equation

$$\sigma = H[\epsilon - \theta - \bar{\xi}\bar{\theta}(\lambda)] \quad (1b)$$

and equilibrium equation

$$\sigma \cdot \delta \epsilon = [q + \bar{\xi}\bar{q}(\lambda)] \cdot \delta u \quad (1c)$$

where

$$\delta \epsilon = L_1(\delta u) + L_{11}(u, \delta u) \quad (2)$$

The external mechanical and thermal loads, q and θ , are assumed to be of the form†

†It should be noted that the results of this section do not rely on the linear dependence of the dead mechanical and thermal loads on λ . The results are unaltered if $\lambda\theta_0$ and λq_0 are replaced by the arbitrary functions $\theta_0(\lambda)$ and $q_0(\lambda)$, respectively. As seen in the next section, in the modified structure method the effective applied loads on the perfect structure are quadratic in λ .

$$\begin{aligned} q &= \lambda[q_0 + q_1(u)] \\ \theta &= \lambda\theta_0 \end{aligned} \quad (3)$$

where $q_1(u)$ (the live load part) is a linear operator satisfying for any two admissible displacement vectors u and v the reciprocal relation

$$q_1(u) \cdot v = q_1(v) \cdot u \quad (4)$$

The perfect structure ($\bar{\xi} = 0$) is assumed to have a bifurcation at load level λ_c from a fundamental prebuckling state $u_0(\lambda)$. The displacement expansion for the imperfect structure is assumed in the form [cf. eq. (31) of ref. 8]

$$\begin{aligned} u &= u_0(\lambda) + \xi u_1 + \xi^2 u_2 + \dots + \bar{\xi}[u_{01} + (\lambda - \lambda_c)u_{01}' + \\ &+ (1/2)(\lambda - \lambda_c)^2 u_{01}'' + \dots + \xi u_{11} + \xi^2 u_{21} + \dots] + O(\bar{\xi}^2) \end{aligned} \quad (5)$$

The prebuckling state $u_0(\lambda)$ is expanded in a Taylor series about $\lambda = \lambda_c$ as follows

$$u_0(\lambda) = u_0^* + (\lambda - \lambda_c)u_0^{(1)*} + (1/2)(\lambda - \lambda_c)^2 u_0^{(2)*} + \dots \quad (6)$$

where (*) denotes that a quantity is evaluated at $\lambda = \lambda_c$. Similar expansions are obtained for $\sigma_0(\lambda)$, $\bar{q}(\lambda)$ and $\bar{\theta}(\lambda)$. As in reference 8, the dependence of λ on ξ and $\bar{\xi}$ is given approximately by

$$(\lambda - \lambda_c)\xi - a\lambda_c \xi^2 - b\lambda_c \xi^3 = -\bar{\xi}[\alpha\lambda_c + \beta(\lambda - \lambda_c) + \gamma\lambda_c \xi] \quad (7)$$

Substituting equation (5) into equation (1a) and using equations (6) and (7) give the strain expansion

$$\begin{aligned} \epsilon &= \epsilon_0(\lambda) + \xi \epsilon_1 + \xi^2 \epsilon_2 + \xi^3 \epsilon_3 + \dots + \bar{\xi}[\epsilon_{01} + (\lambda - \lambda_c)\epsilon_{01}' + \dots \\ &+ \xi \epsilon_{11} + \dots] + O(\bar{\xi}^2) \end{aligned} \quad (8)$$

where expressions for ϵ_1 , ϵ_2 , and ϵ_3 are given in reference 9, and

$$\epsilon_{01} = L_1(u_{01}) + L_{11}(u_0^*, u_{01}) - \alpha\lambda_c L_{11}(u_0^{(1)*}, u_1) \quad (9a)$$

$$\begin{aligned} \epsilon_{01}' &= L_1(u_{01}') + L_{11}(u_0^*, u_{01}') + L_{11}(u_0^{(1)*}, u_{01}) \\ &- \beta L_{11}(u_0^{(1)*}, u_1) - (1/2)\alpha\lambda_c L_{11}(u_0^{(2)*}, u_1) \end{aligned} \quad (9b)$$

$$\begin{aligned} \epsilon_{11} &= L_1(u_{11}) + L_{11}(u_0^*, u_{11}) + L_{11}(u_1, u_{01}) \\ &- \gamma\lambda_c L_{11}(u_0^{(1)*}, u_1) - \alpha\lambda_c L_{11}(u_0^{(1)*}, u_2) \\ &- (1/2)a\alpha\lambda_c^2 L_{11}(u_0^{(2)*}, u_1) \end{aligned} \quad (9c)$$

Equations (9) may be obtained from equations (36) of reference 8 by setting the geometric imperfection (denoted by \bar{U} in that paper) to zero.

Substitution of equation (8) into equation (1b) and using the Taylor series expansion for $\bar{\theta}(\lambda)$ gives the stress expansion

$$\begin{aligned}\sigma = & \sigma_0(\lambda) + \xi\sigma_1 + \xi^2\sigma_2 + \xi^3\sigma_3 + \dots + \bar{\xi}[\sigma_{01} \\ & + (\lambda - \lambda_c)\sigma'_{01} + \dots + \xi\sigma_{11} + \dots] + O(\bar{\xi}^2)\end{aligned}\quad (10)$$

where

$$\sigma_1 = H(\epsilon_1), \sigma_2 = H(\epsilon_2), \sigma_3 = H(\epsilon_3), \dots \quad (11a)$$

$$\sigma_{01} = H(\epsilon_{01} - \bar{\theta}^*), \sigma'_{01} = H(\epsilon'_{01} - \bar{\theta}^{(1)*}), \dots \quad (11b)$$

$$\sigma_{11} = H(\epsilon_{11}), \dots \quad (11c)$$

Substitution of equations (2) and (10) into equation (1c), using equation (7) and the Taylor series expansions for $u_0(\lambda)$, $\sigma_0(\lambda)$, and $\bar{q}(\lambda)$, and equating to zero coefficients of ξ^k , $\bar{\xi}(\lambda - \lambda_c)^k$, $\xi\xi^k$ for $k = 0, 1, 2, \dots$ yields a set of variational equilibrium equations satisfied by the stress components. The equations for σ_1 , σ_2 , and σ_3 are given in reference 9. In addition one obtains

$$\begin{aligned}\sigma_{01} \cdot \delta\epsilon_0^* + \sigma_0^* \cdot L_{11}(u_{01}, \delta u) - \lambda_c q_1(u_{01}) \cdot \delta u - \alpha \lambda_c E^{(1)}(u_1, \delta u) \\ - \bar{q}^* \cdot \delta u = 0\end{aligned}\quad (12a)$$

$$\begin{aligned}\sigma'_{01} \cdot \delta\epsilon_0^* + \sigma_0^* \cdot L_{11}(u'_{01}, \delta u) - \lambda_c q_1(u'_{01}) \cdot \delta u - \bar{q}^{(1)*} \cdot \delta u \\ - \beta E^{(1)}(u_1, \delta u) - (1/2)\alpha \lambda_c E^{(2)}(u_1, \delta u) \\ + E^{(1)}(u_{01}, \delta u) = 0\end{aligned}\quad (12b)$$

$$\begin{aligned}\sigma_{11} \cdot \delta\epsilon_0^* + \sigma_0^* \cdot L_{11}(u_{11}, \delta u) - \lambda_c q_1(u_{11}) \cdot \delta u + \sigma_{01} \cdot L_{11}(u_1, \delta u) \\ + \sigma_1 \cdot L_{11}(u_{01}, \delta u) - \gamma \lambda_c E^{(1)}(u_1, \delta u) - \alpha \lambda_c E^{(1)}(u_2, \delta u) \\ - (1/2)a\alpha \lambda_c^2 E^{(2)}(u_1, \delta u) = 0\end{aligned}\quad (12c)$$

As in references 8 and 9, compatibility conditions for the above field equations yield expressions for the coefficients a , b , α , β , and γ in equation (7). The postbuckling coefficients a and b do not depend on the imperfection and are given by (ref. 9)

$$a = -3\sigma_1 \cdot L_2(u_1) / 2\lambda_c F^{(1)}(u_1, u_1) \quad (13a)$$

$$\begin{aligned}b = -[\sigma_2 \cdot L_2(u_1) + 2\sigma_1 \cdot L_{11}(u_1, u_2) + a\lambda_c F^{(1)}(u_1, u_2) \\ + (1/2)a^2\lambda_c^2 F^{(2)}(u_1, u_1)] / \lambda_c F^{(1)}(u_1, u_1)\end{aligned}\quad (13b)$$

The remaining coefficients, which determine the structural behavior in the presence of a load imperfection, are given by

$$\alpha = -[\sigma_1 \cdot \bar{\theta}^* + \bar{q}^* \cdot u_1] / \lambda_c F^{(1)}(u_1, u_1) \quad (14a)$$

$$\beta = \{F^{(1)}(u_{01}, u_1) - \sigma_1 \cdot \bar{\theta}^{(1)*} - \bar{q}^{(1)*} \cdot u_1 - (1/2)\alpha \lambda_c F^{(2)}(u_1, u_1)\} / F^{(1)}(u_1, u_1) \quad (14b)$$

$$\gamma = -[3\alpha \lambda_c F^{(1)}(u_1, u_2) + 2a \lambda_c F^{(1)}(u_1, u_{01}) + 2\bar{q}^* \cdot u_2 + 2\sigma_2 \cdot \bar{\theta}^* + (1/2)a \alpha \lambda_c^2 F^{(2)}(u_1, u_1)] / \lambda_c F^{(1)}(u_1, u_1) \quad (14c)$$

Once the expansion states and coefficients are calculated, equations (5) and (7) give through the parameter ξ the approximate behavior of the imperfect structure at loads in the vicinity of λ_c . If $\alpha = \beta = 0$, equation (7) has the two solutions

$$\xi = 0 \quad (15a)$$

$$\text{and} \quad \lambda = \lambda_c (1 - \bar{\xi} \gamma + a \xi + b \xi^2) \quad (15b)$$

Bifurcation occurs at the intersection of the two solutions, so that the buckling load λ_s of the imperfect structure is

$$\lambda_s \approx \lambda_c (1 - \bar{\xi} \gamma) \quad (16)$$

In Appendix A, the validity of equation (16) is verified by treating the special case of a load imperfection proportional to the applied load. If $\alpha \neq 0$, the imperfect structure may buckle at a limit load which is found from the condition $d\lambda/d\xi = 0$. To a first approximation, neglecting β and γ , this gives

$$\lambda_s \approx \lambda_c [1 - 2(-\alpha a \bar{\xi})^{1/2}] \quad \text{if } \alpha a \bar{\xi} < 0 \quad (17a)$$

$$\lambda_s \approx \lambda_c [1 + 3(\alpha^2 b \bar{\xi}^2 / 4)^{1/3}] \quad \text{if } a = 0, b < 0 \quad (17b)$$

Fictitious Loads

Given the field equations of the structure, equations (1a), (1b), and (1c) with $\bar{\xi} = 0$, additional loads $\Delta\theta$ and Δq are sought such that a solution of the field equations is the solution of the linearized set of equations given below

$$\varepsilon = L_1(u) \quad (18a)$$

$$\sigma = H(\varepsilon - \lambda\theta_0) \quad (18b)$$

$$\sigma \cdot L_1(\delta u) = \lambda q_0 \cdot \delta u \quad (18c)$$

The solution of these equations depends linearly on the load factor λ , i.e.,

$$u = \lambda \hat{u}; \quad \sigma = \lambda \hat{\sigma} \quad (19)$$

where \hat{u} , $\hat{\sigma}$ represent the unit load ($\lambda = 1$) solution of equations (18). To make the linear solution satisfy the field equations, additional loads have to be applied to the structure. Substitution of u, σ from equation (19) into the field equations, elimination of ε and comparison with equations (18) shows that $\lambda \hat{u}$, $\lambda \hat{\sigma}$ are a solution if the additional loads are given by

$$\Delta\theta = (1/2)\lambda^2 L_2(\hat{u}) \quad (20a)$$

$$\Delta q \cdot \delta u = \lambda^2 [\hat{\sigma} \cdot L_{11}(\hat{u}, \delta u) - q_1(\hat{u}) \cdot \delta u] \quad (20b)$$

Equations (20) determine the additional (fictitious) load system that has to be imposed in order to enforce the linear state.[†] Thus the loads $q + \Delta q$, $\theta + \Delta\theta$ are viewed as the loads of the perfect structure.

The fictitious loads $\Delta\theta$, Δq have the characteristics that: (1) they are second-degree in the load factor λ , and (2) they are dead loads, i.e., they do not depend on the response u . Consequently, they do not enter explicitly into the eigenvalue or higher order equations.

The identification of the linear approximation as the exact response under the loads $q + \Delta q$ and $\theta + \Delta\theta$ permits the retrieval of the response under the actual set of loads by using Koiter's method. This is done by viewing the structure under the actual loads as an imperfect version of the structure under the modified loads. Thus the behavior of the structure is obtained approximately by treating, with the analysis of the previous section, the following load imperfections

$$\bar{\theta} = -\Delta\theta \quad (21a)$$

$$\bar{q} = -\Delta q \quad (21b)$$

with the scalar amplitude $\bar{\xi} = 1$.

[†]Note that whereas the stress and displacement functions obtained from the linearized equations are made exact by the fictitious loads given by equations (20), the strains are not.

It may be noted that since the prebuckling response of the perfect structure is linear, the eigenvalue equations and the equations for higher order terms may be simplified. For example, the term $F^{(2)}(u_1, u_1)$ in equations (14b) and (14c) for β and γ is identically zero. Also, in contrast to the formulation of references 6 and 7, in the present formulation the buckling problem for the perfect structure reduces to a linear eigenvalue problem, since both σ_0 and u_0 are linear functions of λ . From reference 9, these eigenvalue equations are

$$\epsilon_1 = L_1(u_1) + \lambda_c L_{11}(\hat{u}, u_1) \quad (22a)$$

$$\sigma_1 = H(\epsilon_1) \quad (22b)$$

$$\begin{aligned} \sigma_1 \cdot L_1(\delta u) + \lambda_c \sigma_1 \cdot L_{11}(\hat{u}, \delta u) + \lambda_c \hat{\sigma} \cdot L_{11}(u_1, \delta u) \\ - \lambda_c q_1(u_1) \cdot \delta u = 0 \end{aligned} \quad (22c)$$

In summary, to apply the method, the linear prebuckling, eigenvalue, and higher order equations are solved. Using these response states and the imperfection given by equations (21), the coefficients a , b , α , β , and γ are evaluated. The critical load computed from equation (7) with $\bar{\xi} = 1$ is then the desired approximation to the buckling load.

RESULTS

As noted in the Introduction, the purpose of this study was to assess the feasibility of applying the modified structure method to shell structures by actual application to several cases of shells of revolution under axisymmetric loads. For such problems, there are two possible types of elastic instability. For thin shells, the buckling is usually in the form of bifurcation into an asymmetric (harmonic) mode. For thicker shells, axisymmetric limit load buckling may be the failure mode. Both of these cases are treated.

The solution of the linear prebuckling, eigenvalue, and postbuckling equations required in the method was accomplished through the use of prebuckling (ref. 10), buckling (ref. 11), and postbuckling (refs. 5 and 12) computer programs.† The specialization of equations (14) and (20) for the imperfection coefficients and fictitious loads to shells of revolution under axisymmetric loads is given in Appendix B.

†Since the writing of references 5 and 10-12, these programs have been upgraded by the implementation of the Zarghamee method (ref. 13) of solution of one-dimensional boundary value problems.

Sample Shell Calculations

Calculations of nonlinear buckling loads by the modified structure method were made for the following sample shells, which have been studied previously by conventional methods.

- (1) Clamped shallow spherical cap under uniform dead pressure, $\Lambda = 6$ (refs. 11 and 14). The shell subtends a 20° half-angle, and has the properties $\nu = 1/3$, and $t/R = 0.01094$.
- (2) Prolate spheroid under uniform live and dead pressure (ref. 11). The properties of this shell are $\nu = 0.4$, $r_a/r_b = 3$, and $t/r_b = 0.0630$.
- (3) Clamped cylinder under uniform axial compression (ref. 15). The properties of this shell are $\nu = 0.3$, $r/t = 100$, and $l/r = 0.7$.
- (4) Cylinder with edge rings under uniform axial compression (ref. 16). The cylinder properties are identical to the clamped cylinder above, but instead of having clamped edges, it has zero eccentricity square edge rings of thickness $tc^{1/2}$. The clamped cylinder is thus equivalent to this case with $c \rightarrow \infty$.
- (5) Clamped shallow spherical cap under uniform dead pressure, $\Lambda = 4$ (ref. 14). This shell has the thickness ratio $t/R = 0.02462$ but is otherwise the same as (1), above.

The first four cases are bifurcation cases, whereas the fifth case is a limit load case. In the table below calculated values of γ (in bifurcation cases) and $\Gamma = 2(-\alpha a)^{1/2}$ (in the limit load case) are compared to actual relative changes in buckling load due to nonlinearity.

| Case | $1 - \lambda_s/\lambda_c$ | γ | Γ | % err |
|--|---------------------------|----------|----------|-------|
| Spherical cap ($\Lambda = 6$, $n = 2$) | 0.232 | 2.23 | -- | 863 |
| Prolate spheroid (dead pressure, $n = 3$) | 0.00048 | 0.000483 | -- | 0. |
| Prolate spheroid (live pressure, $n = 3$) | -0.00447 | -0.00444 | -- | 0.7 |
| Clamped cylinder ($n = 9$) | 0.161 | 0.932 | -- | 479 |
| Ring-stiffened cylinder ($c = 0.5$, $n = 9$) ^a | -0.00433 | -0.00462 | -- | 6.4 |
| Ring-stiffened cylinder ($c = 1$, $n = 9$) ^a | -0.00636 | -0.00753 | -- | 18.5 |
| Ring-stiffened cylinder ($c = 2$, $n = 9$) ^a | -0.00591 | -0.00905 | -- | 53.0 |
| Spherical cap ($\Lambda = 4$, $n = 0$) | 0.472 | -- | 0.914 | 93.6 |

^aThe values shown for these shells were inadvertently based on nonfundamental eigenvalues.

Since the load imperfection theory predicts that to first order, $1 - \lambda_s/\lambda_c = \gamma$ in bifurcation cases and $1 - \lambda_s/\lambda_c = \Gamma$ in limit load cases, the percent error shown in the table is indicative of the accuracy of the modified structure method. Except in the cases of very small nonlinear effect, this error is disappointingly large. In the case of the cylinder with edge rings, the nonlinearity in the prebuckling state, and hence the size of the load imperfection, approach zero as the ring parameter c approaches zero. Since the percent error in the method also gets small with c , it would appear that the reason for the poor accuracy shown in cases of significant nonlinearity is simply that the load imperfections implied by the method are too large for the first-order theory. The possibility of an error in the numerical calculations was essentially eliminated by independent verification of α and γ by hand, using alternate formulas for these quantities in cases of dead loading, derived in Appendix C.

Estimate of Size of Load Imperfection

The above results suggest that, in many cases, the load imperfection (i.e., the negative of the fictitious loads) is too large relative to the effective applied loads (i.e., real plus fictitious loads) for first-order load imperfection theory to treat accurately. In the case of dead loading, the imperfection exists only in the meridional free thermal strain θ_1 and the surface moment L_2 [eqs. (B-8)]. Within the accuracy of the Donnell-Mushtari-Vlasov approximate shell equations (ref. 17), it can be shown that L_2 is equivalent to a normal surface force X_3 according to $X_3 = (1/r)\partial(rL_2)/\partial s$. Using this equivalence, the local ratio of the imperfection in L_2 to the effective applied pressure at bifurcation is plotted in figure 1 for the spherical cap cases of the previous section.[†] As seen from this figure, the mechanical load imperfection is as great as 83% of the effective applied pressure in the $\Lambda = 4$ cap, and as great as 40% in the $\Lambda = 6$ cap. Although no error estimates exist for the load imperfection analysis, intuitively one might expect imperfections of such magnitudes to be too large for a first-order theory. Also, it is observed that the error in the bifurcation ($\Lambda = 6$) case is much greater than in the limit load ($\Lambda = 4$) case (see previous section), in spite of the fact that the imperfection is greater in the latter case.

Parametric Studies

In order to prove the correctness of the theory and to assess the growth of error with the size of load imperfection, two parametric studies were made. One study was made for the bifurcation case and the other for the limit load case. If the size of the load imperfection is reduced

[†]No such equivalence exists for the thermal load θ_1 ; however, the effect of the mechanical load imperfection is dominant in these problems.

sufficiently, say, by the factor $\bar{\xi}$, from that required to fully remove the fictitious loads, then equations (16) and (17a) should agree with nonlinear buckling loads calculated directly with the additional loads equal to $1 - \bar{\xi}$ times the full fictitious loads.

These calculations were made for the $\Lambda = 6$ and $\Lambda = 4$ spherical caps, and the results are tabulated in tables I and II and plotted in figures 2-5. In table I, the numbers in parentheses are the buckling loads predicted by equation (16), and the nonlinear prebuckling states upon which the bifurcation pressures p_s are based were obtained at these loads. Note from tables I and II that the critical pressures $p_s(0)$ with the full linearizing loads ($\bar{\xi} = 0$) do not exactly agree with the linear result shown in the table captions. This occurs since the additional loads input to the nonlinear prebuckling program themselves contain an unavoidable imperfection due to round-off and truncation errors, as they were computed to at most five significant digits from the results of the linear calculations. Assuming that this numerical imperfection has negligible effect on the values of γ and Γ , $p_s(0)$ is used in place of p_c as the perfect shell buckling pressure in this evaluation.

It is apparent from these results that first-order load imperfection theory handles the limit load case much better than the bifurcation case. However, even in this case the error is too large for application of the modified structure method, which corresponds to $\bar{\xi} = 1$. It is interesting to note that in the limit load case, except for very small imperfections, the error is approximately given by the predicted value of $1 - p_s/p_s(0)$ (fig. 3), which is in agreement with the analytical result presented in Appendix A for a load imperfection proportional to the applied load. In contrast to this, the error in the bifurcation case is roughly 100 times the exact value of $1 - p_s/p_s(0)$ (fig. 2).

CONCLUDING REMARKS

An assessment of the accuracy of the modified structure method as applied to shells has been made by comparing results obtained from it with exact nonlinear buckling loads for several axisymmetric shells under axisymmetric loading. The following conclusions may be inferred from this study.

- (1) The error in the modified structure method as applied here is due to the error in the load imperfection analysis used. This analysis is used to remove the fictitious loads that enforce the linear prebuckling state.

- (2) In cases of bifurcation buckling, except for shells for which the nonlinear effect is so small as to be uninteresting, the fictitious loads are too large to be treated by load imperfection theory.
- (3) The error in load imperfection theory, and therefore in the modified structure method, is considerably smaller in limit load cases than in bifurcation cases. However, even in the limit load case studied, the fictitious loads are too large to be treated accurately by first-order load imperfection theory.

Therefore, use of the modified structure method as an inexpensive means to calculate nonlinear buckling loads of shell structures does not appear to be practical. However, since in the case of general asymmetric loading (or a general shell) the limit load case is far more prevalent than the bifurcation case, it is recommended that further study be conducted to determine:

- (1) if a second-order load imperfection analysis significantly reduces the error for the limit load case studied here, and
- (2) if the present case (a shallow spherical cap) requires unusually large fictitious loads.

APPENDIX A

CONSISTENCY CHECK OF LOAD IMPERFECTION ANALYSIS

Consider the case of a load imperfection proportional to the applied load, which is assumed to be dead [i.e., $q_1(u) \equiv 0$ in eq. (3)]

$$\bar{q}(\lambda) = \lambda q_0 \quad (\text{A-1a})$$

$$\bar{\theta}(\lambda) = \lambda \theta_0 \quad (\text{A-1b})$$

From equations (1) it is clear that such an imperfection is simply equivalent to a change in λ to $\lambda(1 + \bar{\xi})$ in the perfect structure equations. Hence, it follows that if the perfect structure bifurcates at the load λ_c , then the imperfect structure bifurcates at $\lambda_s = \lambda_c / (1 + \bar{\xi})$. For small imperfections, i.e., $|\bar{\xi}| \ll 1$, one thus obtains the result

$$\lambda_s \approx \lambda_c (1 - \bar{\xi} + \bar{\xi}^2) \quad (\text{A-2})$$

Comparison of equation (A-2) with equation (16) suggests that in this case equations (14) should yield $\alpha = \beta = 0$ and $\gamma = 1$, in which event the relative error in equation (16), i.e., $(1 - \lambda_s/\lambda_c - \xi\gamma)/(1 - \lambda_s/\lambda_c)$, is approximately $\bar{\xi}$. If direct calculation of α , β , and γ for this imperfection gives $\alpha = \beta = 0$ and $\gamma = 1$, this will constitute a check on the load imperfection analysis.

Calculation of α

In this case, the numerator of the expression for α [eq. (14a)] is $\lambda_c(\sigma_1 \cdot \theta_0 + q_0 \cdot u_1)$, which is shown below to be zero. The prebuckling field equations for this case are

$$\varepsilon_0 = L_1(u_0) + (1/2)L_2(u_0)$$

$$\sigma_0 = H(\varepsilon_0 - \lambda \theta_0) \quad (\text{A-3})$$

$$\sigma_0 \cdot \delta \varepsilon_0 = \lambda q_0 \cdot \delta u$$

Differentiation of equation (A-3) with respect to λ gives

$$\varepsilon_0^{(1)} = L_1(u_0^{(1)}) + L_{11}(u_0, u_0^{(1)}) \quad (\text{A-4a})$$

$$\sigma_0^{(1)} = H(\varepsilon_0^{(1)} - \theta_0) \quad (\text{A-4b})$$

$$\sigma_0^{(1)} \cdot \delta \varepsilon_0 + \sigma_0 \cdot L_{11}(u_0^{(1)}, \delta u) = q_0 \cdot \delta u \quad (\text{A-4c})$$

Evaluation of equations (A-4) at $\lambda = \lambda_c$ and $\delta u = u_1$ give

$$\varepsilon_0^{(1)*} = L_1(u_0^{(1)*}) + L_{11}(u_0^*, u_0^{(1)*}) \quad (\text{A-5a})$$

$$\sigma_0^{(1)*} = H(\varepsilon_0^{(1)*} - \theta_0) \quad (\text{A-5b})$$

$$\sigma_0^{(1)*} \cdot \varepsilon_1 + \sigma_0^* \cdot L_{11}(u_0^{(1)*}, u_1) = q_0 \cdot u_1 \quad (\text{A-5c})$$

Setting $\delta u = u_0^{(1)*}$ in the eigenvalue equations [eq. (8c) of ref. 9] and using equation (A-5a) gives

$$\sigma_1 \cdot \varepsilon_0^{(1)*} + \sigma_0^* \cdot L_{11}(u_1, u_0^{(1)*}) = 0 \quad (\text{A-6})$$

Substitution of equation (A-6) into (A-5c) gives, in view of the relation $L_{11}(u, v) = L_{11}(v, u)$,

$$q_0 \cdot u_1 = \sigma_0^{(1)*} \cdot \varepsilon_1 - \sigma_1 \cdot \varepsilon_0^{(1)*} \quad (\text{A-7})$$

Substitution of equation (A-5b) into equation (A-7) gives, in view of the symmetry of the linear operator H and equations (11a), the desired result

$$q_0 \cdot u_1 + \sigma_1 \cdot \theta_0 = 0 \quad (\text{A-8})$$

In effect, equation (A-8), and hence $\alpha = 0$, is the compatibility condition for the equations for $u_0^{(1)*}$, since the homogeneous form of equations (A-4), evaluated at $\lambda = \lambda_c$, is satisfied by the buckling mode u_1 .

Calculation of β

From equations (A-1) and (A-8) and the fact that $\alpha = 0$, equation (14b) for β reduces to

$$\beta = F^{(1)}(u_{01}, u_1) / F^{(1)}(u_1, u_1) \quad (\text{A-9})$$

For the imperfection given by equation (A-1), the equations for u_{01} are, from equations (9a), (11b), and (12a),

$$\begin{aligned} \varepsilon_{01} &= L_1(u_{01}) + L_{11}(u_0^*, u_{01}) \\ \sigma_{01} &= H(\varepsilon_{01} - \lambda_c \theta_0) \\ \sigma_{01} \cdot \delta \varepsilon_0^* + \sigma_0^* \cdot L_{11}(u_{01}, \delta u) &= \lambda_c q_0 \cdot \delta u \end{aligned} \quad (\text{A-10})$$

Comparison of equation (A-10) with (A-4) shows that

$$u_{01} = \lambda_c u_0^{(1)*} + C_{01} u_1 \quad (A-11)$$

In this case, it is clear from equation (5) that if $\xi = 0$ is to designate the prebuckling path, then the constant C_{01} must be chosen to be zero. Since an imperfection of size ξ is equivalent to changing the load factor λ by the amount $\xi\lambda$, to first order the prebuckling state u_0^* has the corresponding change $\xi\lambda_c u_0^{(1)*}$, as indicated by equations (5) with $\xi = 0$ and (A-11) with $C_{01} = 0$.

In order to show that $\beta = 0$, from equations (A-9) and (A-11) it is sufficient to show that $F^{(1)}(u_0^{(1)*}, u_1) = 0$. This equality follows from the compatibility of the equations for $u_0^{(2)*}$. To see this, differentiate equation (A-4) with respect to λ and evaluate at $\lambda = \lambda_c$ to give

$$\epsilon_0^{(2)*} = L_1(u_0^{(2)*}) + L_{11}(u_0^*, u_0^{(2)*}) + L_2(u_0^{(1)*}) \quad (A-12a)$$

$$\sigma_0^{(2)*} = H(\epsilon_0^{(2)*}) \quad (A-12b)$$

$$\sigma_0^{(2)*} \cdot \delta \epsilon_0^* + \sigma_0^* \cdot L_{11}(u_0^{(2)*}, \delta u) + 2\sigma_0^{(1)*} \cdot L_{11}(u_0^{(1)*}, \delta u) = 0 \quad (A-12c)$$

Evaluation of equation (A-12c) at $\delta u = u_1$ gives

$$\sigma_0^{(2)*} \cdot \epsilon_1 + \sigma_0^* \cdot L_{11}(u_0^{(2)*}, u_1) + 2\sigma_0^{(1)*} \cdot L_{11}(u_0^{(1)*}, u_1) = 0 \quad (A-13)$$

Setting $\delta u = u_0^{(2)*}$ in the eigenvalue equation [eq. (8c) of ref. 9] and using equation (A-12a) gives

$$\sigma_1 \cdot \epsilon_0^{(2)*} + \sigma_0^* \cdot L_{11}(u_1, u_0^{(2)*}) - \sigma_1 \cdot L_2(u_0^{(1)*}) = 0 \quad (A-14)$$

In view of equations (11a) and (A-12b), $\sigma_1 \cdot \epsilon_0^{(2)*} = \sigma_0^{(2)*} \cdot \epsilon_1$, so that subtraction of equation (A-14) from (A-13) gives

$$\sigma_1 \cdot L_2(u_0^{(1)*}) + 2\sigma_0^{(1)*} \cdot L_{11}(u_0^{(1)*}, u_1) = 0 \quad (A-15)$$

However, from the definition of $F^{(1)}(u, v)$ [see eqs. (15) and (20) of ref. 9], it may be seen that the left-hand side of equation (A-15) is $F^{(1)}(u_0^{(1)*}, u_1)$, which proves that $\beta = 0$.

Calculation of γ

Since $\alpha = 0$ for the imperfection given by equation (A-1), equation (14c) for γ reduces to, in view of equation (A-11)

$$\gamma = -2[a\lambda_c F^{(1)}(u_1, u_0^{(1)*}) + q_0 \cdot u_2 + \sigma_2 \cdot \theta_0] / F^{(1)}(u_1, u_1) \quad (A-16)$$

To calculate the numerator of equation (A-16), evaluate equation (A-4c) at $\lambda = \lambda_c$ and $\delta u = u_2$, and introduce ε_2 [by means of eq. (13a) of ref. 9] to give

$$\sigma_0^{(1)*} [\varepsilon_2 - (1/2)L_2(u_1) - a\lambda_c L_{11}(u_0^{(1)*}, u_1)] + \sigma_0^* \cdot L_{11}(u_0^{(1)*}, u_2) = q_0 \cdot u_2 \quad (A-17)$$

Setting $\delta u = u_0^{(1)*}$ in the variational equation for u_2 [eq. (13c) of ref. 9] and using equation (A-4a) gives

$$\begin{aligned} \sigma_2 \cdot \varepsilon_0^{(1)*} + \sigma_0^* \cdot L_{11}(u_2, u_0^{(1)*}) + \sigma_1 \cdot L_{11}(u_1, u_0^{(1)*}) \\ + a\lambda_c E^{(1)}(u_1, u_0^{(1)*}) = 0 \end{aligned} \quad (A-18)$$

Subtraction of equation (A-18) from (A-17) gives

$$q_0 \cdot u_2 + \sigma_2 \cdot \theta_0 = -(1/2)F^{(1)}(u_1, u_1) - a\lambda_c F^{(1)}(u_1, u_0^{(1)*}) \quad (A-19)$$

Substitution of equation (A-19) into (A-16) gives the desired result, $\gamma = 1$.

APPENDIX B

SHELLS OF REVOLUTION

In this appendix, the fictitious loads of the modified structure method are derived for shells of revolution. These are then specialized to the case of axisymmetric loading. For this case the pertinent coefficients in the $\lambda - \xi$ relation are written in terms of shell variables.

Fictitious Loads

Based on moderate rotation theory of shells (see, e.g., ref. 11) the strain operator $L_2(u)$ in equation (1a) is

$$L_2(u) \equiv \begin{Bmatrix} \chi^2 + \omega^2 \\ \psi^2 + \omega^2 \\ 2\chi\psi \end{Bmatrix} \quad (B-1)$$

where the three rows correspond to the stretching strains ϵ_1 , ϵ_2 , and ϵ_{12} , respectively. Since the curvature-displacement relations are linear, $L_2(u)$ has no components corresponding to the bending strains κ_1 , κ_2 , and κ_{12} . Similarly, for rings one has

$$L_2(u) = \omega_x^2 + \omega_y^2 \quad (B-2)$$

In this case, the single element of $L_2(u)$ corresponds to the hoop strain ϵ_ϕ .

From equations (20a), (B-1), and (B-2), the fictitious free thermal strains are, for the shell,

$$\begin{aligned} \Delta\theta_1 &= (1/2)\lambda^2(\hat{\chi}^2 + \hat{\omega}^2) \\ \Delta\theta_2 &= (1/2)\lambda^2(\hat{\psi}^2 + \hat{\omega}^2) \\ \Delta\theta_{12} &= \lambda^2\hat{\chi}\hat{\psi} \end{aligned} \quad (B-3)$$

and, for rings,

$$\Delta\theta_R = (1/2)\lambda^2(\hat{\omega}_x^2 + \hat{\omega}_y^2) \quad (B-4)$$

Note that the fictitious free thermal strains do not vary through the shell or ring thickness.

In order to derive the mechanical fictitious loads, it is noted that the term $\sigma \cdot L_{11}(u, \delta u)$ [cf. eq. (20b)] in the equation of virtual work becomes, after partial integration, precisely the virtual work of the nonlinear terms in the shell equations of equilibrium and natural boundary conditions [eqs. (4) and (6) of ref. 11]. These equations have been transformed in reference 5 by replacing the shell stress resultants by the natural shell force variables P, Q, S along undeformed axial, radial, and circumferential directions, thereby eliminating the nonlinear terms in the natural boundary conditions and, consequently, the need for fictitious line loads applied at shell edges.[†] From the identification of the nonlinear terms in the transformed equations as pseudo loads [eqs. (12) or ref. 5] and the relation between \tilde{T}_{12} and S [eq. (14a) of ref. 5], one obtains the fictitious shell loads

$$\begin{aligned}\Delta X_1 &= \lambda^2 \{ [(\hat{T}_1 + \hat{T}_2)\hat{\omega}]^* / r - p\hat{\chi} \} \\ \Delta X_2 &= \lambda^2 \{ (r'/r)(\hat{T}_1 + \hat{T}_2)\hat{\omega} - p\hat{\psi} \} \\ \Delta X_3 &= -\lambda^2 [p(\hat{e}_1 + \hat{e}_2) + \hat{u} \cdot \nabla p] \\ \Delta L_1 &= -\lambda^2 [\hat{T}_2\hat{\psi} + \hat{S}\hat{\chi}] + \lambda^3 [(\hat{T}_1 + \hat{T}_2)\hat{\chi}\hat{\omega}/2] \\ \Delta L_2 &= \lambda^2 [\hat{T}_1\hat{\chi} + \hat{S}\hat{\psi}] - \lambda^3 [(\hat{T}_1 + \hat{T}_2)\hat{\psi}\hat{\omega}/2]\end{aligned}\tag{B-5}$$

where

$$\tilde{S} = S - 2\tilde{M}_{12}/R_2\tag{B-6}$$

In equations (B-5) the terms in p represent a normal pressure field, if one exists, as required by the term $q_1(\hat{u}) \cdot \delta u$ in equation (20b). The cubic terms in λ arise from the linearization of S instead of \tilde{T}_{12} , and probably are negligible since they are smaller than similar terms by a factor of the rotation ω . It is also noted that the quadratic nonlinearity of \tilde{T}_{12} [see eq. (14a) of ref. 5] introduces a very small, if not negligible, quadratic nonlinearity in the eigenvalue in the associated buckling equations.

[†]The transformation from \tilde{T}_{12} to S introduces an additional small nonlinearity in the constitutive relations. This nonlinearity can be eliminated by introducing a fictitious thermal stress $\Delta\theta_{12} = (1/2)\lambda^2(\hat{T}_1 + \hat{T}_2)\hat{\omega}$. An imperfection thermal stress $\bar{\theta}_{12} = -\Delta\theta_{12}$ is then required to compensate for $\Delta\theta_{12}$, and corresponding to this, terms of the form $\sigma_k \cdot \bar{\theta}^{(p)*}$ in equations (14) should each be augmented by the work of $\bar{\theta}_{12}^{(p)*}$, viz. $\iint (\partial \bar{\theta}_{12}^{(p)*} / \partial \lambda^p) \epsilon_{12}^{(k)} r d\phi ds$, where the derivative is evaluated at $\lambda = \lambda_c$ and $\epsilon_{12}^{(k)}$ is the shear strain of the buckling mode ($k = 1$) or the second-order postbuckling state ($k = 2$).

For rings one obtains, in a similar way, the fictitious loads

$$\begin{aligned}\Delta F_x &= \Delta F_y = \Delta F_\phi = \Delta N_\phi = 0 \\ \Delta N_x &= \lambda^2 \hat{T}_\phi \hat{\omega}_x \\ \Delta N_y &= \lambda^2 \hat{T}_\phi \hat{\omega}_y\end{aligned}\tag{B-7}$$

Axisymmetric Loading

In the special case of axisymmetric loading considered in this study, the fictitious loads are simplified greatly. Writing the load imperfection components as the negative of the fictitious loads, one has from equations (B-3) and (B-5)

$$\begin{aligned}\bar{\theta}_1 &= -(1/2)\lambda^2 \hat{\chi}^2 \\ \bar{X}_1 &= \lambda^2 p \hat{\chi} \\ \bar{X}_3 &= \lambda^2 [p(\hat{e}_1 + \hat{e}_2) + \xi \partial p / \partial x + \hat{n} \partial p / \partial y] \\ \bar{L}_2 &= -\lambda^2 \hat{T}_1 \hat{\chi} \\ \bar{\theta}_2 &= \bar{\theta}_{12} = \bar{X}_2 = \bar{L}_1 = 0\end{aligned}\tag{B-8}$$

Also from equations (B-4) and (B-7), no fictitious ring loads are required and hence there is no ring load imperfection.

Asymmetric bifurcation. - In this case, $a = 0$; also, since the load imperfection is axisymmetric and the buckling mode harmonic, it follows from equation (14a) that $\alpha = 0$. Equations (9a), (11b), and (12a) then show that u_{01} is axisymmetric, and so from equation (14b) it follows that $\beta = 0$. The corrected bifurcation load λ_s is then given to first order in terms of the linear bifurcation load λ_c by equation (16) (with $\bar{\xi} = 1$) and γ is calculated from the simplified form of equation (14c)

$$\gamma = -2(\bar{q}^* \cdot u_2 + \sigma_2 \cdot \bar{\theta}^*) / \lambda_c F^{(1)}(u_1, u_1)\tag{B-9}$$

In terms of shell variables, the numerator in equation (B-9) is

$$\bar{q}^* \cdot u_2 + \sigma_2 \cdot \bar{\theta}^* = 2\pi \int (\bar{X}_1^* u + \bar{X}_3^* w + \bar{L}_2^* \chi + {}_0T_1 \bar{\theta}_1^*) r ds\tag{B-10}$$

The denominator $F^{(1)}(u_1, u_1)$ has been given as equation (B-3) of reference 14.

Axisymmetric limit load.- In this case, the limit load λ_s is calculated to first order from the axisymmetric linear bifurcation load λ_c through equation (17a) (with $\bar{\xi} = 1$). The coefficient a is calculated from equation (13a) and α is calculated from equation (14a). In terms of shell variables, the numerator in equation (13a) is

$$\sigma_1 \cdot L_2(u_1) = 2\pi T_1 \chi^2 r ds \quad (B-11)$$

where T_1 and χ are buckling mode variables. The numerator in equation (14a) is identical in form to that given in equation (B-10) with the displacements and stress resultant of the axisymmetric component of the u_2 -state being replaced by the corresponding buckling mode variables. Also, in this case, $F^{(1)}(u_1, u_1)$ is twice the value given by equation (B-3) of reference 14.

APPENDIX C

ALTERNATE FORMULAS FOR LOAD IMPERFECTION PARAMETERS

In the modified structure method the effective applied load contains both a part which depends linearly on the load factor λ (the real load) and a part which is second-degree in λ (the fictitious load). Consider a load imperfection which is proportional by the factor $\bar{\xi}$ to the sum of the linear part $\lambda(q_0, \theta_0)$ (dead loading is assumed), plus twice the second-degree part $\Delta q, \Delta \theta = \lambda^2(q_f, \theta_f)$, i.e.,

$$\begin{aligned}\bar{q}(\lambda) &= \lambda q_0 + 2\lambda^2 q_f \\ \bar{\theta}(\lambda) &= \lambda \theta_0 + 2\lambda^2 \theta_f\end{aligned}\tag{C-1}$$

Such an imperfection is equivalent to the new loading, $(1 + \bar{\xi})\lambda(q_0, \theta_0) + (1 + 2\bar{\xi})\lambda^2(q_f, \theta_f)$. Insofar as $1 + 2\bar{\xi} \approx (1 + \bar{\xi})^2$ for small $\bar{\xi}$, this is equivalent to simply replacing λ by $\lambda(1 + \bar{\xi})$ in the original loading. Hence, for sufficiently small $\bar{\xi}$, it follows that if the perfect structure bifurcates at the load λ_c , the imperfect structure bifurcates at

$$\lambda_s \approx \lambda_c / (1 + \bar{\xi}) \approx \lambda_c (1 - \bar{\xi})\tag{C-2}$$

Comparison of equation (C-2) with equation (16) suggests that in this case, as in the case of pure linear loading and an imperfection proportional to it (Appendix A), $\alpha = \beta = 0$ and $\gamma = 1$. Verification of these results from equations (14) follows along the same lines as the calculation of α , β , and γ in Appendix A and is not repeated here.

In the modified structure method, the imperfection treated is the negative of the fictitious load $\Delta q, \Delta \theta$ [eqs. (21)]. Formulas for α and γ for the imperfection $\bar{q} = 2\Delta q$, $\bar{\theta} = 2\Delta \theta$ can be derived from the above results simply by taking the contribution of the linear imperfection of equation (C-1) to the right-hand side of equations (14), viz.

$$\alpha = (\sigma_1 \cdot \theta_0 + q_0 \cdot u_1) / F^{(1)}(u_1, u_1)\tag{C-3a}$$

and if $\alpha = a = 0$

$$\gamma = 1 + 2(\sigma_2 \cdot \theta_0 + q_0 \cdot u_2) / F^{(1)}(u_1, u_1)\tag{C-3b}$$

Since the imperfection of the modified structure method differs from this imperfection by the factor $-1/2$, one obtains from equations (C-3) alternate formulas for α and γ of the method, viz.

$$\alpha = -(\sigma_1 \cdot \theta_0 + q_0 \cdot u_1) / 2F^{(1)}(u_1, u_1) \quad (C-4a)$$

and if $\alpha = a = 0$

$$\gamma = -1/2 - (\sigma_2 \cdot \theta_0 + q_0 \cdot u_2) / F^{(1)}(u_1, u_1) \quad (C-4b)$$

Note that these formulas are free of explicit dependence on the fictitious loads, q_0 and θ_0 being the real applied mechanical load and free thermal strain at unit λ . They provide an independent verification of the numerical evaluation of α and γ , including the calculation of the prerequisite prebuckling, buckling, and postbuckling states.

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TABLE I

CLAMPED SPHERICAL CAP, $\Lambda = 6$, $p_c/E \times 10^5 = 14.8572$, $n_c = 2$

| $\bar{\xi}$ | $\bar{\xi}_\gamma$ | $p_s/E \times 10^5$ | $\bar{\xi}_\gamma^* = 1 - p_s/p_s(0)$ | % error = $(\gamma/\gamma^* - 1) \times 100$ |
|-------------|--------------------|---------------------|---------------------------------------|--|
| 0 | 0 | 14.85780 (14.86) | 0 | -- |
| 0.0001 | 0.000223 | 14.85455 (14.85) | 0.000219 | ~2 |
| .001 | .00223 | 14.8298 (14.82) | .00188 | 19 |
| .004 | .00894 | 14.7717 (14.72) | .00579 | 54 |
| .01 | .0223 | 14.690 (14.53) | .0113 | 98 |
| .02 | .0447 | 14.593 (14.19) | .0178 | 150 |
| .05 | .1118 | 14.400 (13.20) | .0308 | 263 |

TABLE II

CLAMPED SPHERICAL CAP, $\Lambda = 4$, $p_c/E \times 10^5 = 81.220$, $n_c = 0$

| $\bar{\xi}$ | $\bar{\xi}^{1/2}_\Gamma$ | $p_s/E \times 10^5$ | $\bar{\xi}^{1/2}_\Gamma^* = 1 - p_s/p_s(0)$ | % error = $(\Gamma/\Gamma^* - 1) \times 100$ |
|-------------|--------------------------|---------------------|---|--|
| 0 | 0 | 81.302 | 0 | -- |
| 0.001 | 0.02890 | 78.964 | 0.02876 | 0.5 |
| .01 | .0914 | 74.45 | .0842 | 8.6 |
| .1 | .2890 | 63.20 | .2226 | 29.8 |
| 1 | .914 | 42.9 | .472 | 93.6 |

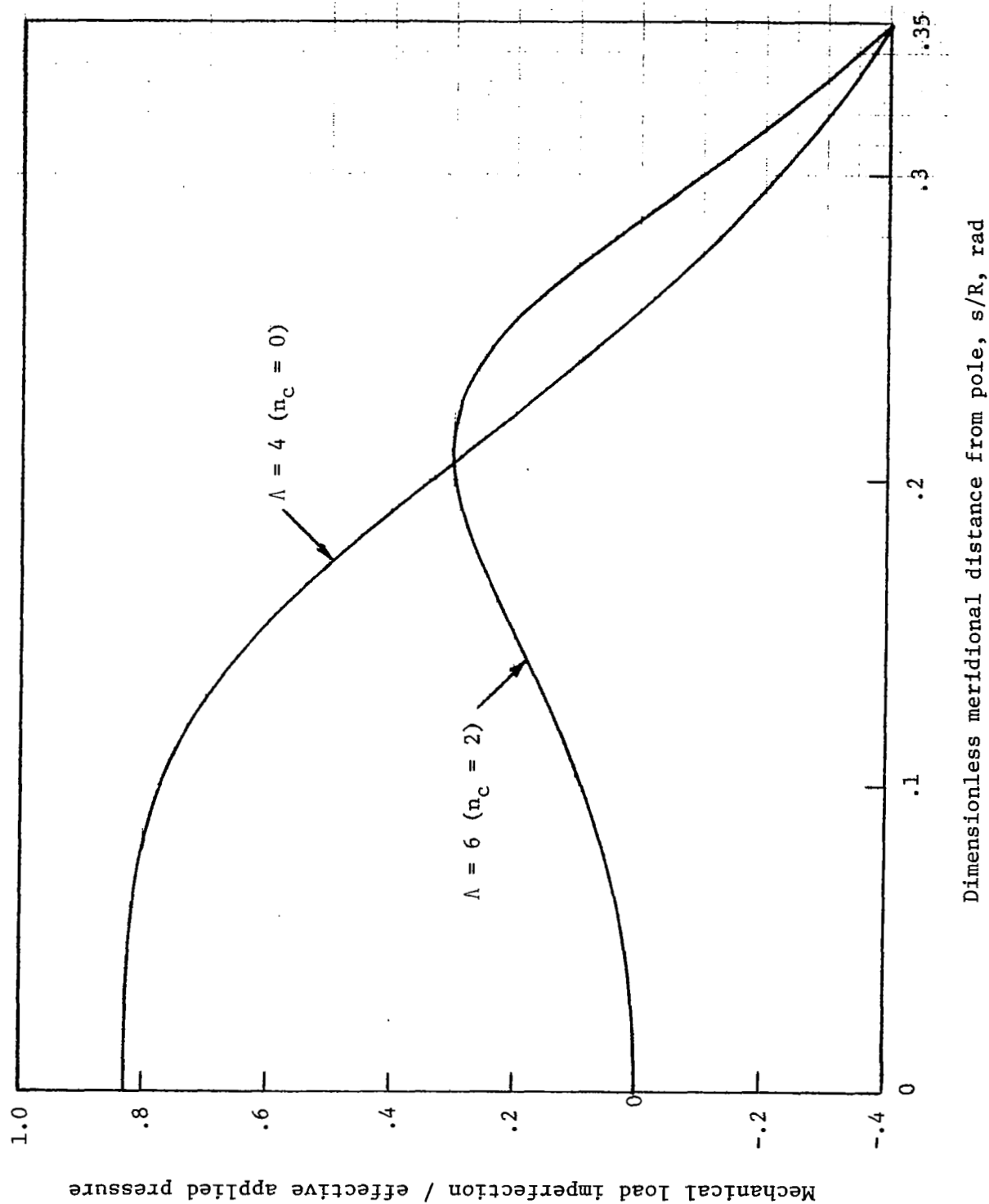


FIGURE 1. MECHANICAL LOAD IMPERFECTION AT BIFURCATION
OF CLAMPED SHALLOW SPHERICAL SHELLS

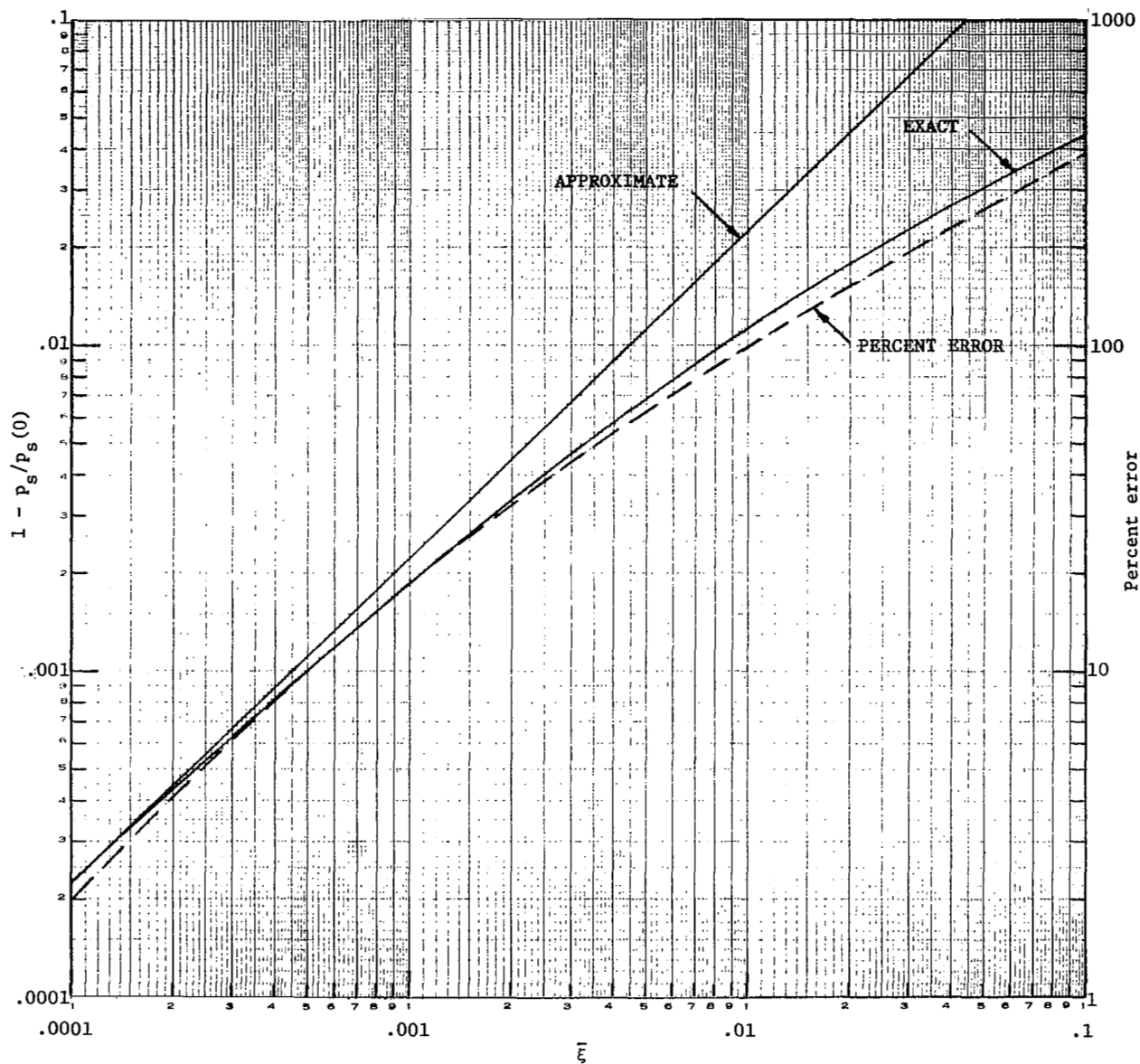


FIGURE 2. RELATIVE CHANGE IN CRITICAL PRESSURE OF CLAMPED SHALLOW SPHERICAL SHELLS ($\lambda = 6$) VS. FRACTION OF FICTITIOUS LOADS REMOVED

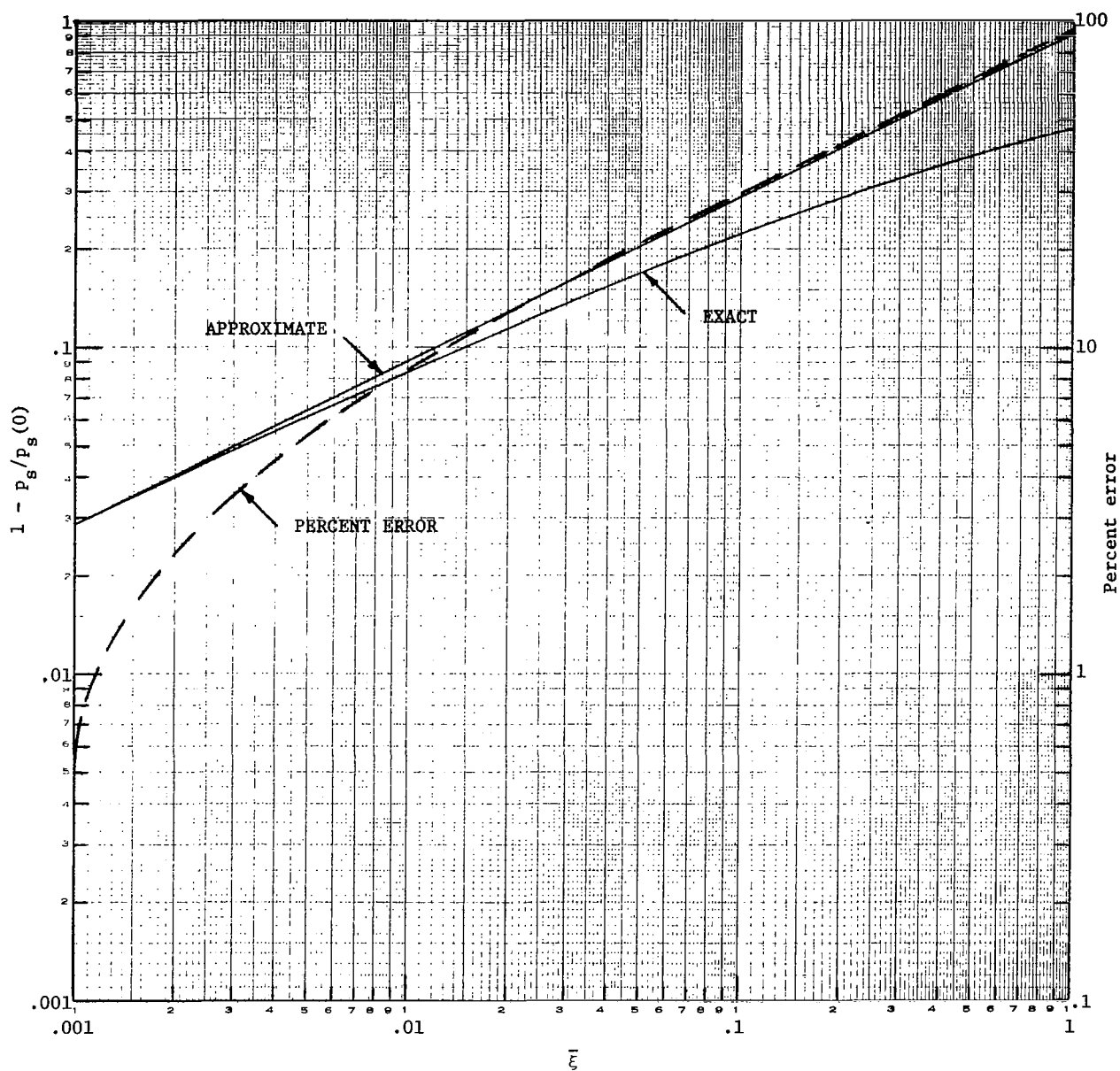


FIGURE 3. RELATIVE CHANGE IN CRITICAL PRESSURE OF CLAMPED SHALLOW SPHERICAL SHELLS ($\lambda = 4$) VS. FRACTION OF FICTITIOUS LOADS REMOVED

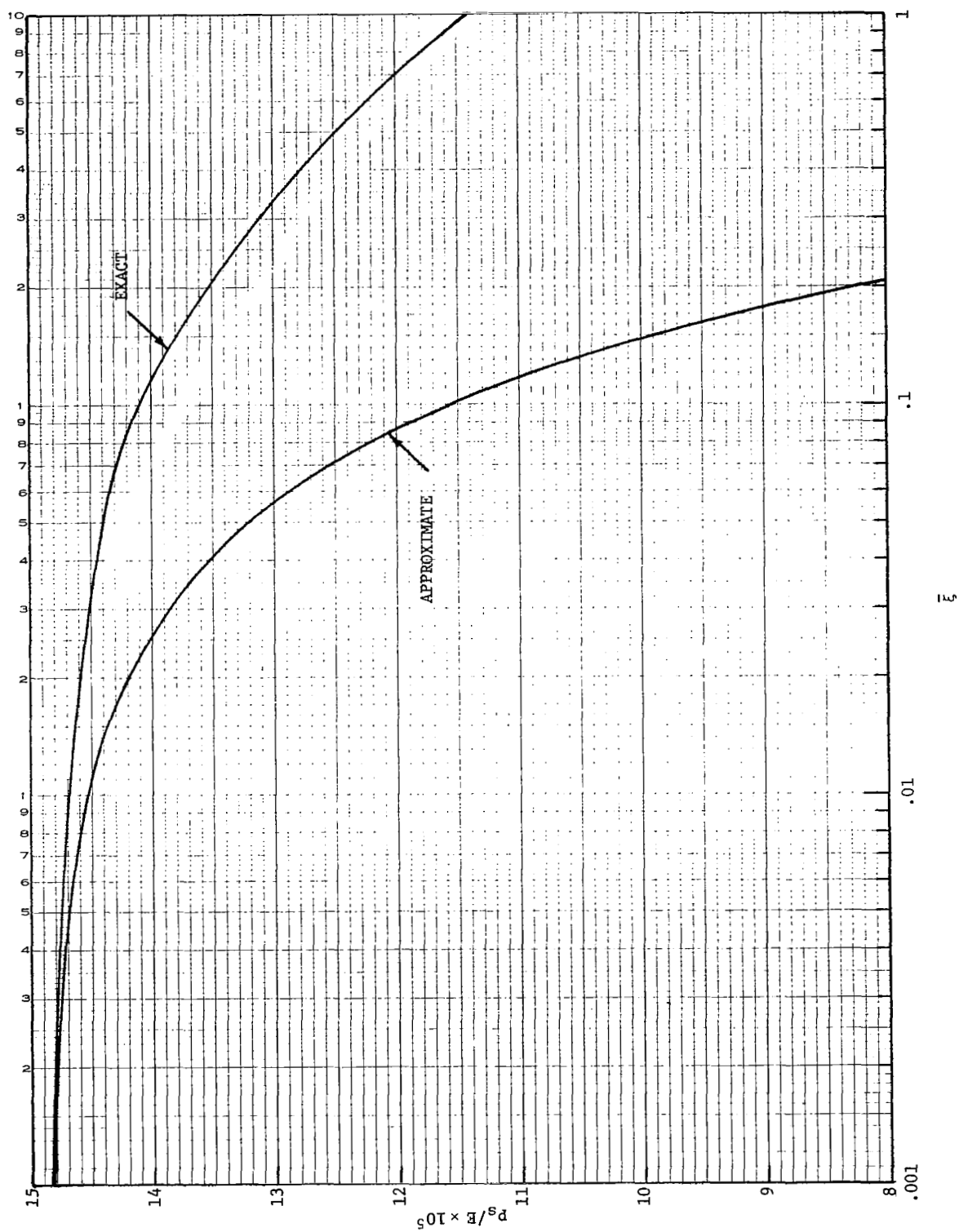


FIGURE 4. CRITICAL PRESSURE OF CLAMPED SHALLOW SPHERICAL SHELLS ($\lambda = 6$)
VS. FRACTION OF FICTITIOUS LOADS REMOVED

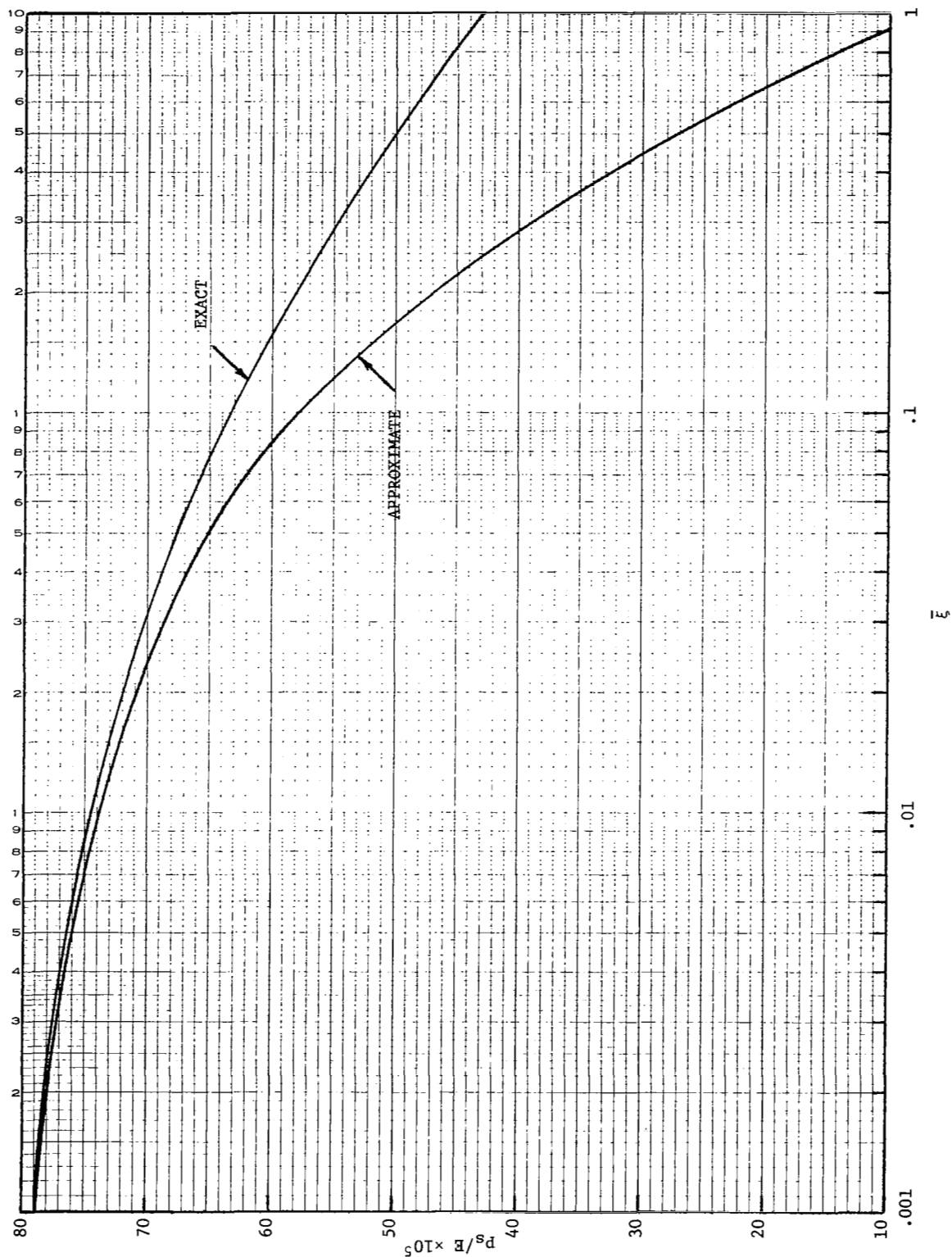


FIGURE 5. CRITICAL PRESSURE OF CLAMPED SHALLOW SPHERICAL SHELLS ($\lambda = 4$)
VS. FRACTION OF FICTITIOUS LOADS REMOVED